THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Solution of Tutorial Classwork 7

1. (a) Suppose not. Then for any $i_1, i_2, \ldots, i_n \in I$, we have $\cap_{k=1}^n F_{i_k} \not\subset U$. This implies that $\cap_{k=1}^n (F_{i_k} \cap (X \backslash U)) \neq \emptyset$ for any $i_1, i_2, \ldots, i_n \in I$. Note that since F_i's and $X \backslash U$ are closed, $(F_i \cap (X \backslash U))$ are closed for all $i \in I$. Hence, by Finite Intersection Property (FIP), we have $\cap_{i \in I}(F_i \cap (X \backslash U)) \neq \emptyset$. However, this implies that $\cap_{i \in I} F_i \not\subset U$, contradiction. Hence there exists $i_1, i_2, \ldots, i_n \in I$ such that $\cap_{k=1}^n F_{i_k} \subset U$.

   (b) In this case, we consider the topological space $(F_{i_0}, \mathfrak{T}|_{F_{i_0}})$. Note that by assumption, $(F_{i_0}, \mathfrak{T}|_{F_{i_0}})$ is a compact topological space. Since F_i's are closed, $F_i \cap F_{i_0}$'s are also closed. Since U is open, $U \cap F_{i_0}$ is also open. Since $\cap_{i \in I} F_i \subset U$, we have $\cap_{i \in I}(F_i \cap F_{i_0}) \subset (U \cap F_{i_0})$. Hence, by a), we know that there exists $i_1, i_2, \ldots, i_n \in I$ such that $\cap_{k=1}^n (F_{i_k} \cap F_{i_0}) \subset (U \cap F_{i_0})$. This implies that $F_{i_0} \cup (\cap_{k=1}^n F_{i_k}) \subset (U \cap F_{i_0}) \subset U$.

2. (a) Note that for any $i_1 < i_2 < \cdots < i_n$, we have $\cap_{k=1}^n F_{i_k} = F_{i_n} \neq \emptyset$. Hence, by FIP, $\cap_{n \in \mathbb{N}} F_n \neq \emptyset$.

   (b) * Consider the sequence $F_1 = X, F_{n+1} = f(F_n)$. Since X is a compact Hausdorff space, f is a closed map (i.e. f maps closed sets to closed sets). So by induction, F_n is a collection of non-empty closed sets.

   Now we are going to show that $F_{n+1} \subset F_n$ for $n \geq 1$. First of all, since $F_2 = f(X) \subset X = F_1$, the proposition is true for $n = 1$. Assume it is true for some $k \in \mathbb{N}$, i.e. $F_{k+1} \subset F_k$. Then we have $f(F_{k+1}) \subset f(F_k)$. Hence $F_{k+2} \subset F_{k+1}$. By induction, we have $F_{n+1} \subset F_n$ for $n \geq 1$.

   As a result, by a), we have $\cap_{n \in \mathbb{N}} F_n \neq \emptyset$. Let $F = \cap_{n \in \mathbb{N}} F_n \neq \emptyset$. We are going to show that $f(F) = F$.

   First, we have $f(F) = f(\cap_{n \in \mathbb{N}} F_n) \subset \cap_{n \in \mathbb{N}} f(F_n) = \cap_{n \in \mathbb{N}} F_{n+1} = F$. To show that $F \subset f(F)$, pick any $y \in F = \cap_{n \in \mathbb{N}} F_n$. Consider the set $K_n = f^{-1}(\{y\}) \cap F_n$. Note that $\{K_n\}_{n \in \mathbb{N}}$ are non-empty closed subsets with $K_{n+1} \subset K_n$ for $n \geq 1$. Hence by a), we have $\cap_{n \in \mathbb{N}} K_n = f^{-1}(\{y\}) \cap (\cap_{n \in \mathbb{N}} F_n) \neq \emptyset$. This implies that $y \in f(F)$ and hence $F = f(F)$.